On the rooted Tutte polynomial

F. Y. Wu,[1] C. King,[2] and W. T. Lu[1]

[1]Department of Physics
[2]Department of Mathematics
Northeastern University
Boston, Massachusetts 02115

February 1, 2008

Abstract

The Tutte polynomial is a generalization of the chromatic polynomial of graph colorings. Here we present an extension called the rooted Tutte polynomial, which is defined on a graph where one or more vertices are colored with prescribed colors. We establish a number of results pertaining to the rooted Tutte polynomial, including a duality relation in the case that all roots reside around a single face of a planar graph. The connection with the Potts model is also reviewed.

1 The Tutte polynomial

Consider a finite graph G with vertex set V and edge set E. A spanning subgraph $G'(S) \subseteq G$ is a subgraph of G containing all members of V and an edge set $S \subseteq E$. Let C be a set of q distinct colors. A q-coloring of G is a coloring of the vertices in V such that two vertices connected by an edge bear different colors. It is well-known that the number of q-colorings of G is given by the chromatic polynomial [1]

$$P(G;q) = \sum_{S \subseteq E} q^{p(S)} (-1)^{|S|}, \tag{1}$$

where $p(S)$ is the number of components in the spanning subgraph $G'(S)$. Alternately, we can regard (1) as generating colorings of components of spanning subgraphs of G with q colors with an edge weight -1.

As an extension of the chromatic polynomial, Tutte [2, 3, 4] introduced what is now known as the Tutte polynomial

$$Q(G;t,v) = \sum_{S \subseteq E} t^{p(S)} v^{|S|-|V|+p(S)}. \tag{2}$$

Indeed, one has the relation

$$P(G;q) = (-1)^{|V|} Q(G;-q,-1). \tag{3}$$

In view of (3), it is useful to write (2) as

$$Q(G;t,v) = v^{-|V|} \sum_{S \subseteq E} (vt)^{p(S)} v^{|S|}, \tag{4}$$

so that for $vt = q =$ positive integers, the Tutte polynomial (4) generates colorings of components of spanning subgraphs of G with q colors and edge weights v, instead of $v = -1$.

For planar G with dual graph G_D, it is well-known that the Tutte polynomial possesses the duality relation

$$v\ Q(G;t,v) = t\ Q(G_D;v,t), \tag{5}$$

a relation first observed by Whitney [5].

2 The rooted Tutte polynomial

We extend the definition (4) to a rooted Tutte polynomial.

A vertex is rooted, or is a root, if it is colored with a prescribed (fixed) color. A graph is rooted if it contains rooted vertices. Let R denote a set of n roots located at vertices $\{r_1, r_2, \cdots, r_n\}$. A *color configuration* is a map $x : R \longmapsto C$, and as a convenient shorthand we write $x(r_i) = x_i$ for $i = 1, 2, \cdots, n$. A component of a spanning subgraph is *exterior* if it contains one or more roots, and is *interior* otherwise. An exterior component is *proper* if all roots in the component are of the same color. A spanning subgraph $G'(S)$ is proper if all its exterior components are proper. An edge set $S_x \subseteq E$ is proper if the spanning subgraph $G'(S_x)$ it generates is proper.

For a prescribed color configuration $\{x_1, x_2, \cdots, x_n\}$ of the n roots, we introduce in analogy to (4) the *rooted* Tutte polynomial[1]

$$Q_{x_1 x_2 \cdots x_n}(G; t, v) = v^{-|V|} \sum_{S_x \subseteq E} (vt)^{p_{\mathrm{in}}(S_x)} v^{|S_x|}, \tag{6}$$

where the summation is taken over all proper edge sets S_x, and $p_{\mathrm{in}}(S_x)$ is the number of interior components of $G'(S_x)$. Thus, as in (3), we have for positive integral q the relation

$$\begin{aligned} (-1)^{|V|} Q_{x_1 x_2 \cdots x_n}(G; -q, -1) \;&=\; \text{the number of } q\text{-colorings of } G \text{ with} \\ &\quad\ \text{color configuration } \{x_1, x_2, \cdots, x_n\}. \end{aligned} \tag{7}$$

Clearly, the expression (6) depends on how the n roots are partitioned into subsets of different colors, and the actual colors do not enter the picture.

The coloring configuration $\{x_1, x_2, \cdots, x_n\}$ induces a partition X of R into blocks (subsets) such that all roots in one block are of one color, and colors of different blocks are different. Namely, two elements $r_i, r_j \in R$ belong to the same block of X if and only if they have the same prescribed color $x_i = x_j$. Consider now the summation in (6). Let $G'(S)$ be any (not necessarily proper) spanning subgraph of G. The connected components of $G'(S)$ induce a partition on the set of vertices V of G. We get hence also a partition $\pi(S)$ on the set of rooted vertices R by restricting this partition to R. Clearly, the spanning subgraph $G'(S_x)$ is proper if and only if the

[1] Strictly speaking, it is the expression $v^{|V|} Q_{x_1 x_2 \cdots x_n}(G; t, v)$ which is a polynomial in v and t.

partition $\pi(S_x)$ is a refinement of the partition X. It follow that we can rewrite (6) as

$$Q_X(G;t,v) = \sum_{X' \preceq X} F_{X'}(G;t,v), \tag{8}$$

where

$$F_{X'}(G;t,v) \equiv v^{-|V|} \sum_{S_x \subseteq E,\ \pi(S_x)=X'} (vt)^{p_{\rm in}(S_x)} v^{|S_x|}. \tag{9}$$

Here, we have abbreviated $Q_{x_1 x_2 \cdots x_n}$ by Q_X, which is permitted since the actual colors do not enter the picture at this point. Also it is understood that G is now a rooted graph, with root set R.

The expression (8) assumes the form of a transformation of a partially ordered set. Its inverse is given by the Möbius inversion

$$F_X(G;t,v) = \sum_{X'} \mu(X',X) Q_{X'}(G;t,v), \tag{10}$$

where [6]

$$\begin{aligned} \mu(X',X) &= (-1)^{|X'|-|X|} \prod_{\text{blocks } \epsilon\ X} (n_b(X')-1)!, \qquad \text{if } X' \preceq X \\ &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}, \end{aligned} \tag{11}$$

$n_b(X')$ being the number of blocks of X' that are contained in the block b of X. Note that for $n=1$ we have $|X|=|X'|=1$, $p_{\rm in}(S_x)=p(S_x)-1$, and all edge sets $S \subseteq E$ are proper. Hence we have

$$F_X(G;t,v) = Q_X(G;t,v) = (vt)^{-1} Q(G;t,v), \qquad n=1. \tag{12}$$

This completes the definition and general description of the rooted Tutte polynomial for any graph G.

3 Planar graphs

From here on we consider G being planar with the n roots residing around a single face of G. Without the loss of generality, we can choose the face to be the infinite face and order the roots in the sequence $\{r_1, r_2, \cdots, r_n, r_1\}$ as shown in Fig. 1. A partition X of the n roots is *non-planar* if two roots of one

block separate two roots of another block in the cyclic sequence. Otherwise X is *planar*. For a given n, there are b_n partitions, where [9]

$$b_n = \sum_{m_\nu=0}^{\infty}\Big[n!/\prod_{\nu=1}^{\infty}(\nu!)^{m_\nu}m_\nu!\Big], \qquad \sum_{\nu=1}^{\infty}\nu m_\nu = n, \tag{13}$$

and of the b_n partitions

$$c_n = (2n)!/n!(n+1)! \tag{14}$$

are planar [7, 8]. We shall adopt the convention of writing $X = \{ij, k\ell\cdot\cdot, \cdots\}$ for colors $\{x_i = x_j,\ x_k = x_\ell = \cdot\cdot, \cdots\}$, with $\{ij\}, \{k\ell\cdot\cdot\}, \cdots$ each in order [8]. For example, two partitions for $n = 5$ are

$$\begin{aligned} X_1 &= \{123, 4, 5\}, \qquad |X_1| = 3, \qquad \text{planar}, \\ X_2 &= \{24, 351\}, \qquad |X_2| = 2, \qquad \text{non}-\text{planar}. \end{aligned} \tag{15}$$

Now if G is planar and X' is non-planar then by definition the summand in (9) is empty and one has $F_{X'}(G; t, v) = 0$. Thus we have

Proposition 1:

For planar G

$$F_X(G; t, v) = 0, \qquad \text{if } X \text{ is non}-\text{planar}. \tag{16}$$

This proposition was first established in [9] for the Potts model correlation function (see section 6) by considering its graphical expansion similar to the consideration given in the above. As a consequence of Proposition 1 and the use of (10), we now have

Corollary 1:

Rooted Tutte polynomials associated with non-planar partitions can be written as linear combinations of the rooted Tutte polynomials associated with (refined) planar partitions.

Corollary 1 leads to the sum-rule identities reported in [9] for the Potts correlation function. In the case of $n = 4$, for example, the identity $F_{\{13,24\}}(G; t, v) = 0$ leads to the sum rule [9]

$$Q_{\{13,24\}}(G; t, v) = Q_{\{13,2,4\}}(G; t, v) + Q_{\{1,3,24\}}(G; t, v) - Q_{\{1,2,3,4\}}(G; t, v). \tag{17}$$

From here on we shall restrict our considerations to rooted Tutte polynomials associated with the c_n planar partitions only.

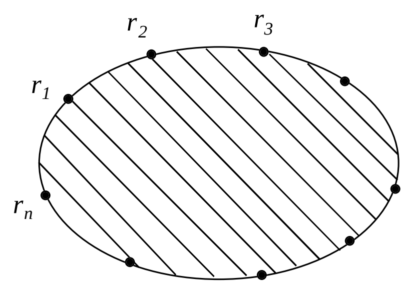

Figure 1: A planar graph G with n roots. The graph is denoted by the shaded region and the n roots by the black circles.

4 The graph G^*

The rooted Tutte polynomial (6) possesses a duality relation for planar graphs, which relates the rooted Tutte polynomial on a graph G to that of a related graph G^*. Here we define G^*. Starting from a planar G, place an extra vertex f in the infinite face and connect it to each root of G by an edge. This gives a new graph G'', which has one more vertex than G and n additional edges. The dual graph of G'' is also planar, and it has a face F containing the extra vertex f. Now remove the n edges on the boundary of F, and the resulting graph is G^*.

It is readily seen that the graph G^* has

$$|V^*| = |V_D| + n - 1 \tag{18}$$

vertices where $|V_D|$ is the number of vertices of G_D, the dual of G, and there is a one-one correspondence between the edges of G and G^*. We denote the set of n vertices $\{r_1^*, r_2^*, \cdots, r_n^*\}$ of G^* surrounding the face F by R^*, with r_i^* residing between the two edges $\langle f, r_{i-1}\rangle$ and $\langle f, r_i\rangle$ of G'', where $r_0 = r_n$. An example of a G and the related G^* for $n = 4$ is shown in Fig. 2. Clearly, the relation of G to G^* is reciprocal, namely, we have $(G^*)^* = G$.

Now each planar partition X of R induces a partition X^* of the set R^* [10]. In order to define X^*, for each block b of X we choose a point in the infinite face of G, and connect all roots r_i in b to this point by drawing new edges. Because X is planar, the points for the blocks can be chosen so that

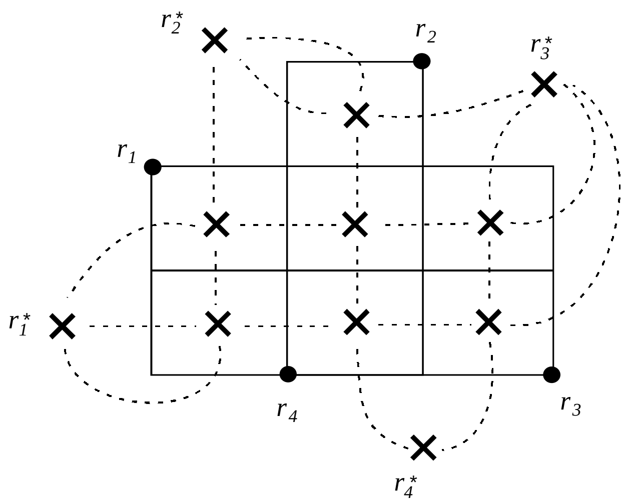

Figure 2: A graph G (solid lines) and the related graph G^* (broken lines) for $n = 4$. Black circles denote roots of G and crosses denote vertices of G^*.

the edges of different blocks do not cross, and the resulting extended graph is still planar. So this process divides the infinite face into regions. The induced partition X^* is then described by the condition that all roots of R^* in one region are regarded as belonging to one block of X^*. Alternately, for another way of defining X^*, let P_{ij}, $1 \le i < j \le n$, be a partition of R into two blocks, the sets $\{r_i, \cdots, r_{j-1}\}$ and $\{r_j, \cdots, r_{i-1}\}$, where all numbers are modulo n. Then, the partition X^* induced by X is defined by the condition that the two roots r_i^* and r_j^* belong to the same block in X^* if and only if X is a refinement of P_{ij}.

We write $X \to X^*$ if X induces X^*. Clearly, X^* is planar, and we have

$$|X| + |X^*| = n + 1. \tag{19}$$

For the planar partition X_1 in (15), for example, we have

$$X_1 = \{123, 4, 5\} \to X_1^* = \{2, 3, 451\}, \qquad |X_1| = |X_1^*| = 3. \tag{20}$$

However, as a result of our labelling convention, the color configuration of the partition $(X^*)^*$ further induced by X^* is a cyclic shift of that of X, namely,

$$\{x_1, x_2, \cdots, x_n\} \to X^* \to \{x_n, x_1, \cdots, x_{n-1}\}. \tag{21}$$

In the example above, for instance, we have

$$\{123, 4, 5\} \to \{2, 3, 451\} \to \{234, 5, 1\}. \tag{22}$$

Finally, there is a one-to-one correspondence between the edge sets E of G and E^* of G^*, an edge set $S \subseteq E$ defines a "complement" edge set $S^* \subseteq E^*$ by the condition that an edge is included in S^* if and only if its corresponding edge is not included in S. Clearly, we have $(S^*)^* = S$.

5 The duality relation

The rooted Tutte polynomial arises in statistical physics as the correlation function of the Potts model (see next section). In a recent paper [10] we have established a duality relation for the Potts correlation function for planar G. However, the proof of the duality relation given in [10] is cumbersome and not easily deciphered in graphical terms. Here we re-state the results as two propositions in the context of the rooted Tutte polynomial, and present direct graph-theoretical proofs of the propositions.

Proposition 2:

For planar G and G^* and the associated planar partitions $X \to X^*$, we have

$$v^{|X|} F_X(G; t, v) = t^{|X^*|} F_{X^*}(G^*; v, t). \tag{23}$$

Proof: Let S_x be a proper edge set on G. We have the Euler relation

$$|S_x| + |S_x^*| = |V| + |V_D| - 2 \tag{24}$$

and, after eliminating n and $|V_D|$ using (18), (19) and (24), the identity

$$|S_x| + |X| - |V| = |S_x^*| + |X^*| - |V^*|, \tag{25}$$

which holds for any proper edge set S_x. Note that we have also the fact

$$\pi(S_x) = X \quad \text{ if and only if } \quad \pi(S_x^*) = X^*. \tag{26}$$

Let $c(S_x^*)$ the number of independent circuits in the spanning subgraph $G'(S_x^*)$. Then we have

$$p(S_x) = c(S_x^*) + |X|. \tag{27}$$

Also, starting from the $|V^*|$ isolated vertices on G^*, one constructs $G'(S_x^*)$ by drawing edges of S_x^* on G^* one at a time. Since each edge reduces the number of components by one except when the adding of an edge completes an independent circuit, one has also

$$p(S_x^*) = |V^*| - |S_x^*| + c(S_x^*). \tag{28}$$

Eliminating $c(S_x^*)$ using (27) and (28) and making use of the relations

$$\begin{aligned} p(S_x) &= p_{\rm in}(S_x) + |X|, \\ p(S_x^*) &= p_{\rm in}(S_x^*) + |X^*|, \end{aligned} \tag{29}$$

one obtains

$$p_{\rm in}(S_x^*) = p_{\rm in}(S_x) + |V^*| - |S_x^*| - |X^*|. \tag{30}$$

The Proposition 2 now follows from the substitution of (30) into the right-hand side of (23) where, explicitly,

$$F_{X^*}(G^*; v, t) = t^{-|V^*|} \sum_{S_x^* \subseteq E^*,\ \pi(S_x^*) = X^*} (vt)^{p_{\rm in}(S_x^*)} t^{|S_x^*|}, \tag{31}$$

and the use of the identities (25) and (26). This completes the proof of Proposition 2.

Proposition 2 was first conjectured in [9] and established later in [10] in the context of Potts correlation functions (see next section) without the explicit reference to the polynomial form (9).

Remark: For $n = 1$, the duality relation (23) for the rooted Tutte polynomial becomes the duality relation (5) for the Tutte polynomial. This is a consequence of (12).

Proposition 3:

1. The rooted Tutte polynomials associated with the c_n planar partitions for G and G^* are related by the duality transformation

$$Q_X(G; t, v) = \sum_Y \mathbf{T}_n(X, Y) Q_Y(G^*; v, t), \tag{32}$$

where $\mathbf{T}_n$ is a $c_n \times c_n$ matrix with elements

$$\mathbf{T}_n(X, Y) = t^{n+1} \sum_{X' \preceq X} (vt)^{-|X'|} \mu(Y, Y'), \qquad X' \to Y'. \tag{33}$$

2. The matrix $\mathbf{T}_n$ satisfies the identity

$$[\mathbf{T}_n]^2(X, X') = \delta(x_1, x_2')\delta(x_2, x_3') \cdots \delta(x_n, x_1'). \tag{34}$$

Proof: The transformation (32) follows by combining (8) and (10) with Proposition 2, and its uniqueness is ensured by the uniqueness of the Möbius inversion. The property (34) is a consequence of (21).

Proposition 3 was first given in [10] in the context of the Potts correlation function (see next section). Explicit expression of $\mathbf{T}_n$ for $n = 2, 3, 4$ can be found in [10] and [11].

6 The Potts and the random cluster models

It is well-known in statistical physics that the Tutte polynomial gives rise to the partition function of the Potts model [12]. In view of the prominent role played by the Potts model in many fields in physics, it is useful to review this equivalence and the further equivalence of the rooted Tutte polynomial with the Potts correlation function.

The q-state Potts model [13] is a spin model defined on a graph G. The spin model consists of $|V|$ spins placed at the vertices of G with each spin taking on q different states and interacting with spins connected by edges. Without going into details of the physics [12] which lead to the Potts model, it suffices for our purposes to define the Potts partition function

$$Z(G; q, v) \equiv \sum_{S \subseteq E} q^{p(S)} v^{|S|}, \tag{35}$$

the n-point partial partition function

$$Z_X(G; q, v) \equiv \sum_{S_x \subseteq E} q^{p_{\rm in}(S_x)} v^{|S_x|}, \tag{36}$$

and the n-point correlation function

$$P_n(G; x_1, x_2, \cdots, x_n) = P_n(G; X) \equiv Z_X(G; q, v)/Z(G; q, v), \tag{37}$$

where again, in analogy to notation in Sections 1 and 2, we have denoted the color configuration $\{x_1, x_2, \cdots, x_n\}$ by the associated partition X. More generally, for any real or complex q, the partition function (35) defines the

random cluster model of Fortuin and Kasteleyn [14], which coincides with the Potts model for integral q.

Relating this to the Tutte polynomial, we now have

$$\begin{aligned} Z(G;q,v) &= v^{|V|}Q(G;t,v) \\ Z_X(G;q,v) &= v^{|V|}Q_X(G;t,v) \\ P_n(G;X) &= Q_X(G;t,v)/Q(G;t,v), \end{aligned} \tag{38}$$

for $q = vt$. The duality relation (5) for the Tutte polynomial then implies the following duality relation for the Potts partition function [10, 13, 15]

$$v^{1-|V|}Z(G;q,v) = (v^*)^{1-|V_D|}Z(G_D;q,v^*), \tag{39}$$

where

$$vv^* = q. \tag{40}$$

One further defines the dual correlation function

$$P_n^*(G^*;X^*) \equiv q\ Z_{X^*}(G^*;q,v^*)/Z(G_D;q,v^*), \tag{41}$$

and also the functions A_X and B_{X^*} by

$$P_n(G;X) = \sum_{X' \preceq X} A_{X'}(G;q,v) \tag{42}$$

and

$$P_n^*(G^*;X^*) = \sum_{X^{*\prime} \preceq X^*} B_{X^{*\prime}}(G^*;q,v^*). \tag{43}$$

Then, Proposition 2 leads to the relation

$$A_X(G;q,v) = q^{-|X|}B_{X^*}(G^*;q,v^*), \qquad X \to X^*. \tag{44}$$

which is the main result of [10].

7 Summary and discussions

We have introduced the rooted Tutte polynomial (6) as a two-variable polynomial associated with a rooted graph and deduced a number of pertinent results.

Our first result is that the rooted Tutte polynomial assumes the form (8) of a partially order set for which the inverse can be uniquely determined. For planar graphs and all roots residing surrounding a single face, we showed that (Proposition 1) the inverse function vanishes for non-planar partitions of the roots. We further showed that the inverse function satisfies the duality relation (23) (Proposition 2) which, in turn, leads to the duality (32) for the rooted Tutte polynomial (Proposition 3). We also reviewed the connection of the Tutte and rooted Tutte polynomials with the Potts model in statistical physics.

Finally, we remark that results reported here have previously been obtained in [9] and [10] in the context of the Potts correlation function. Here, the results are reformulated as properties of the rooted Tutte polynomial and thereby permitting graph-theoretical proofs.

ACKNOWLEDGMENTS

We are grateful to the referee for providing an independent proof of Proposition 1 and numerous suggested improvements on an earlier version of this paper. This work is supported in part by the National Science Foundation grants DMR-9614170 (FYW and WTL) and DMS-9705779 (CK).

References

[1] G. D. Birkhoff, A determinant formula for the number of ways of coloring of a map *Ann. Math.* **14** (1912) 42-46.

[2] W. T. Tutte, A contribution to the theory of chromatic polynomials, *Can. J. Math.* **6** (1954) 80-91.

[3] W. T. Tutte, On dichromatic polynomials, *J. Comb. Theory*, **2** (1967) 301-320.

[4] W. T. Tutte, Graph Theory, in *Encyclopedia of Mathematics and Its Applications*, Vol. 21 (Addison-Wesley, Reading, Massachusetts, 1984), Chap. 9.

[5] H. Whitney, The coloring of graphs, *Ann. Math.*, **33** (1932) 688-718.

[6] See, for example, L. J. van Lint and R. M. Wilson, *A course in combinatorics* (Cambridge University Press, Cambridge, 1992), p. 301.

[7] H. N. V. Temperley and E. H. Lieb, Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattice: some exact results for the percolation problem, *Proc. R. Soc.* London A **322** (1971) 251-280.

[8] W. T. Tutte, The matrix of chromatic joins, *J. Comb. Theory* B **57** (1993) 269-288.

[9] F. Y. Wu and H. Y. Huang, Sum rule identities and the duality relation for the Potts n-point boundary correlation function, *Phys. Rev. Lett.* **79** (1997) 4954-4957.

[10] W. T. Lu and F. Y. Wu, On the duality relation for correlation functions of the Potts model, *J. Phys. A: Math. Gen.* **31** (1998) 2823-2836.

[11] F. Y. Wu, Duality relations for Potts correlation functions, *Phys. Letters* A **228** (1997) 43-47.

[12] See, for example, F. Y. Wu, The Potts Model, *Rev. Mod. Phys.*, **54** (1982) 235-268

[13] R. B. Potts, Some generalized order-disorder transformations, *Proc. Camb. Philos. Soc.* **48** (1954) 106-109.

[14] C. M. Fortuin and P. W. Kasteleyn, On the random-cluster model I. Introduction and relation to other models, *Physica* **57** (1972) 536-564.

[15] F. Y. Wu and Y. K. Wang, Duality transformation in a many-component spin model, *J. Math. Phys.*, **17** (1976) 439-440.